Pressure as Lagrange multiplier—incompressibility and Euler's equations

The incompressibility paradox

It is conceptually difficult, if not impossible, to admit the existence of a solid or fluid which is literally incompressible. Think of a line of particles connected by totally rigid rods; if I apply a force to the particle at the end, there is no way that the particles further down can experience the fact that a force has been applied. The same goes for a three dimensional solid whose bonds are ideally rigid, if an external isotropic stress is applied. In this way, pressure, which is a thermodynamic state variable, has actually no influence on the state of the macroscopic body: either solid or fluid, which presents us with a paradox. Simply put, the bulk modulus cannot be infinite. A popular question in past times that was supposed to puzzle the "man on the Clapham omnibus" was, what happens when an irresistible force meets an immovable object?

On the other hand, incompressibility, $\operatorname{div} \mathbf{v} = 0$, is a cornerstone of fluid dynamics and a postulate embedded in the Navier–Stokes equations as usually stated. How are we to think of pressure in an incompressible fluid? One answer put forward by Arnold Sommerfeld is that *pressure is the Lagrange multiplier in the Hamiltonian mechanics that guarantees that dilatation vanishes in any virtual displacement of the fluid particles.*

The swimmer in trouble

Here is an exciting story. A lifeguard on one of those tall chairs sees a swimmer in distress out to sea and someway off to the right. How can she get to the swimmer in the shortest time? She could take a straight line but she knows that she can run on sand much faster than she can swim. So should she take the shortest possible distance in the water and run to the point perpendicular to the swimmer (point P in figure 1) and enter the water at a right angle to the shore? But then the length of the path she takes is much longer. There is a compromise, which is the bent path as shown in figure 1. The calculation is an easy one. The total distance is the distance on sand, ℓ_s, plus the distance in water, ℓ_w; and if she runs with a speed c_s on sand and c_w in water, then the time to get to the swimmer is,

$$t = \frac{\ell_s}{c_s} + \frac{\ell_w}{c_w}\,.$$

The constants that define the problem are hers and the swimmer's perpendicular distances to the shore, y_s and y_w, and the distance parallel to the shore, x_w, from the swimmer to the point in the water perpendicular to the lifeguard, who luckily knows some trigonometry and some calculus. Using Pythagoras, the distances she has to travel on sand and in water are,

$$\ell_s = \sqrt{y_s^2 + x^2} \qquad \text{and} \qquad \ell_w = \sqrt{(x_w - x)^2 + y_w^2}\,,$$

where x is the distance along the shore from her perpendicular projection to the point at which she enters the water. Using calculus, she seeks the value of x that minimises the time, t:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= 0 \\ &= \frac{1}{c_s}\frac{1}{2}\frac{2x}{\sqrt{y_s^2+x^2}} + \frac{1}{c_w}\frac{1}{2}\frac{-2\,(x_w-x)}{\sqrt{(x_w-x)^2+y_w^2}}\,.\end{aligned}$$

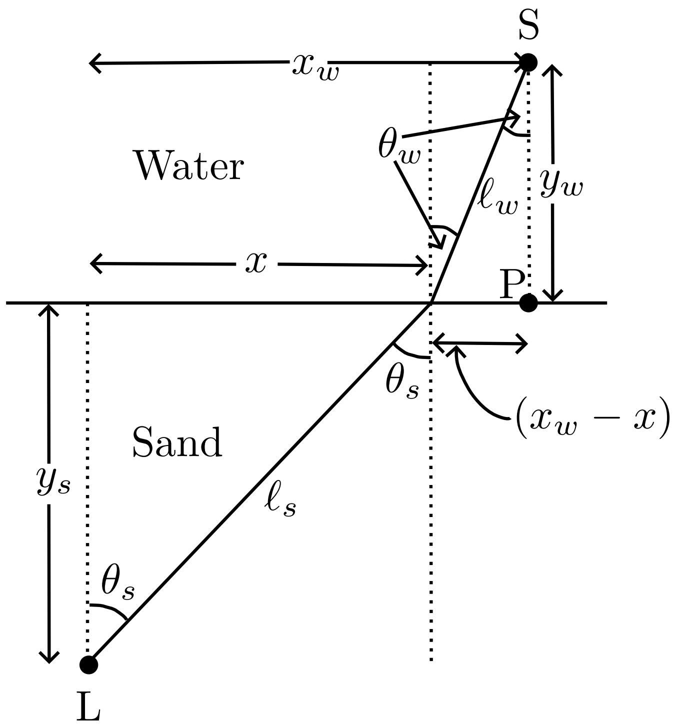

FIGURE 1

This leads to,

$$\frac{1}{c_s}\frac{x}{\sqrt{y_s^2+x^2}} = \frac{1}{c_w}\frac{(x_w-x)}{\sqrt{(x_w-x)^2+y_w^2}}\,,$$

or, simply,

$$\frac{1}{c_s}\frac{x}{\ell_s} = \frac{1}{c_w}\frac{x_w-x}{\ell_w}\,,$$

which she can solve for x. There is a happy ending: she reaches the swimmer in time.

Principle of least time

Another very illuminating way to write the same result is,

$$\frac{\sin\theta_s}{\sin\theta_x} = \frac{c_s}{c_w}\,,$$

and this carries over into Snell's law of refraction of light since the ratio of the speeds of light in two media is the inverse ratio of the refractive indices.

In fact what we have here is an example of Fermat's principle of least time, which is the basis of all geometric optics. The path between the two fixed points, L and S, that takes the least time to travel is that path for which the path integral is stationary,

$$\delta \int_{t_0}^{t_1} \mathrm{d}t = 0\,.$$

The δ indicates a *virtual* displacement of the path; of all the possible paths between the fixed points, the path actually taken is that which renders the path integral minimal, or stationary. This forms the basis, not only of Fermat's formulation in geometric optics but Feynman's scarcely credible formulation of quantum mechanics as a "sum over paths". Here's a quote from Freeman J. Dyson in a statement in 1980, as reported in *Quantum Reality: Beyond the New Physics,* (1987) by Nick Herbert.

> Thirty-one years ago [1949], Dick Feynman told me about his "sum over histories" version of quantum mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave function." I said to him, "You're crazy." But he wasn't.

Principle of least action

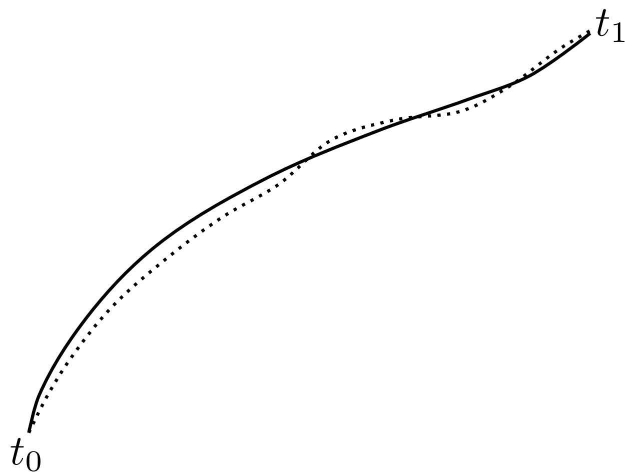

FIGURE 2

Figure 2 shows the trajectory of a particle as it travels from a point at time, t_0, to somewhere else arriving at a time t_1. The particle is experiencing an applied force and possesses a certain kinetic energy, K, by virtue of its mass and velocity. This could indeed be a representation in coordinate space of a collection of particles having positions $\mathbf{r}_k$ and velocities $\dot{\mathbf{r}}_k$. The trajectory (or trajectories) can be discovered using Newton's laws. Alternatively there is a variational principle that may be used, which is contained within Hamilton's formulation of classical mechanics. Rather as in Fermat's principle, *nature chooses that path in configuration space for which the* action *is stationary and minimal.* I don't have the space here to *derive* Hamilton's principle. First you need d'Alembert's principle and then you consider variations of the trajectory under certain conditions. Firstly, as in the Fermat case above, the end points cannot be varied; secondly it is asserted that at any configuration along the path which is varied by a small amount, as indicated in figure 2, if the system is at configuration $\mathbf{r}_k$ in the actual path then its variation to $\mathbf{r}_k + \delta\mathbf{r}_k$ happens at the same time. That is to say, all variations are subject to

$$\delta t = 0\,.$$

Under those two conditions we imagine displacing the path, as in figure 2, by a small amount, entailing a variation of the kinetic energy, δK, and employing an amount of work, δW, against the applied forces. Then Hamilton's principle (which I haven't proved here) is,

$$\int_{t_0}^{t_1} (\delta K + \delta W)\,\mathrm{d}t = 0\,.$$

If the forces are *conservative* then the work derives from a potential energy function, V, and the more usual statement of Hamilton's variational principle is then,

$$\delta \int_{t_0}^{t_1} (K - V)\,\mathrm{d}t = 0\,,$$

and $(K - V)\,\mathrm{d}t$ is the increment of *action.* Kinetic energy take away potential energy is called the Lagrangian, L. The statement is then equivalent to

$$\delta \int_{t_0}^{t_1} L\mathrm{d}t = 0\,.$$

The time integral of the Lagrangian is stationary and minimal for the path that is dictated by nature (Newton's laws).

Pressure as Lagrange multiplier

Finally we can get to the paradox of the incompressible fluid. Here is Sommerfeld's argument. Within a fluid in motion we impose a virtual displacement of certain particles of fluid:

$$\delta\mathbf{u} = \hat{\mathbf{\imath}}\,\delta u_x + \hat{\mathbf{\jmath}}\,\delta u_y + \hat{\mathbf{k}}\,\delta u_z\,.$$

The dilatation is

$$\delta e = \frac{\partial \delta u_x}{\partial x} + \frac{\partial \delta u_y}{\partial y} + \frac{\partial \delta u_z}{\partial z} = \mathrm{div}\,\delta\mathbf{u}\,, \tag{1}$$

which vanishes if the fluid is incompressible. We need to apply Hamilton's principle to this variation; and to ensure that the dilatation vanishes we include the constraint of a Lagrange multiplier, λ,

$$\int_{t_0}^{t_1} \mathrm{d}t \int \mathrm{d}\tau\,(\delta k^v + \delta w^v + \lambda\delta e) = 0\,, \tag{2}$$

in which the second integration is over the volume of the fluid, with volume element, $\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. It is understood that K and W are referred to *unit volume* of fluid (so as in my usual notation, these are rendered in lower case with a superscript, v). δw^v is the so called *virtual work* per unit volume entailed in displacing the trajectory as in figure 2. We will have,

$$\delta w^v = \mathbf{f}^v \cdot \delta\mathbf{u}\,,$$

where $\mathbf{f}^v$ is the force per unit volume; and

$$k^v = \frac{1}{2}\rho v^2 \,,$$

where ρ is the mass density and $\mathbf{v}$ is the velocity field; hence,

$$\delta k^v = \rho \mathbf{v} \cdot \delta \mathbf{v} \,.$$

Furthermore,

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \,,$$

and so its *virtual* variation is,

$$\delta \mathbf{v} = \delta \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta \mathbf{u} \,.$$

Now,

$$\int_{t_0}^{t_1} \rho \mathbf{v} \cdot \delta \mathbf{v} \, \mathrm{d}t = \int_{t_0}^{t_1} \rho \mathbf{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta \mathbf{u} \, \mathrm{d}t = -\int_{t_0}^{t_1} \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \delta \mathbf{u} \, \mathrm{d}t \,,$$

using partial integration, noting that the boundary term vanishes by virtue of the condition that the variation is zero at the end points of the trajectory. That disposes of the first term in the integrand of (2).

For the third term, using (1), we are interested in

$$\int \lambda \, \mathrm{div} \, \delta \mathbf{u} \, \mathrm{d}\tau \,,$$

which can be recast using the derivative of a product rule,

$$\mathrm{div} \, (\lambda \delta \mathbf{u}) = \mathrm{grad} \, \lambda \cdot \delta \mathbf{u} + \lambda \, \mathrm{div} \, \delta \mathbf{u} \,.$$

This leads to,

$$\int \lambda \, \mathrm{div} \, \delta \mathbf{u} \, \mathrm{d}\tau = -\int \mathrm{grad} \, \lambda \cdot \delta \mathbf{u} \, \mathrm{d}\tau + \text{a surface integral.}$$

The surface integral follows from Gauss's divergence theorem and Sommerfeld shows this to vanish. Then Hamilton's integral (2) becomes,

$$\int_{t_0}^{t_1} \mathrm{d}t \int \mathrm{d}\tau \left(-\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{f}^v - \mathrm{grad} \, \lambda \right) \cdot \delta \mathbf{u} = 0 \,.$$

Because the constraint (1) has been taken care of through the Lagrange multiplier, λ, *any* virtual displacement of the trajectory may be made and since thereby $\delta \mathbf{u}$ is arbitrary, the contents of the large parentheses must vanish. This secures Euler's equations (Navier–Stokes for an incompressible inviscid fluid),

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{f}^v - \mathrm{grad} \, p \,,$$

as long as we identify λ with the hydrodynamic, or thermodynamic pressure. Thereby, as Sommerfeld argues, the hydrodynamic pressure in an incompressible fluid is the *reaction force (per unit area) against the constraint of incompressibility.* An analogy is the pendulum whose mass is acted upon by gravity; the Lagrange multiplier that insists the trajectory follows a circle is proportional to the tension, or centripetal plus gravitational force, in the rod or string. Let me show you how this works.

The (non linear) pendulum

The pendulum bob has a mass, m, and a weightless string or rod, of length ℓ. We use Cartesian coordinates with the origin at the pivot, and the y-axis pointing downwards. In order to confine the mass to a circle of radius ℓ, we need $x^2 + y^2 = \ell^2$, so we have this constraint:

$$g(x, y) = x^2 + y^2 - \ell^2 = 0\,.$$

The kinetic energy is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)\,,$$

and the potential energy is

$$V = -mgy\,,$$

because y increases *downwards.* The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy\,,$$

and if I include the constraint I need,

$$\begin{aligned} 0 &= \delta\int_{t_0}^{t_1}\big(L + \lambda g(x, y)\big)\mathrm{d}t \\ &= \delta\int_{t_0}^{t_1}\left(\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy + \lambda\left(x^2 + y^2 - \ell^2\right)\right)\mathrm{d}t \end{aligned}$$

in which λ is the Lagrange multiplier. It must be possible to make the variations in the independent variables, x, y and λ separately. Taking, first, x, we need

$$\delta\int_{t_0}^{t_1}\left(\frac{1}{2}m\dot{x}^2 + \lambda x^2\right)\mathrm{d}t$$

to vanish. Now

$$\frac{1}{2}\delta\left(m\dot{x}^2\right) = m\dot{x}\delta\dot{x} = m\dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\delta x\,.$$

I can integrate that kinetic energy term by parts and the boundary terms will vanish by definition of the path, however it is varied, having fixed end points at t_0 and t_1:

$$\int_{t_0}^{t_1} m\dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\delta x\,\mathrm{d}t = -\int_{t_0}^{t_1} m\ddot{x}\delta x\,\mathrm{d}t\,.$$

The remaining constraint term is

$$\delta\left(\lambda x^2\right) = 2\lambda x\delta x\,,$$

and therefore the required variation with respect to x is,

$$\int_{t_0}^{t_1} (-m\ddot{x} + 2\lambda x)\, \delta x \, \mathrm{d}t\,.$$

This must hold for an arbitrary variation in x, δx, and hence the contents of the parentheses vanish: $m\ddot{x} = 2\lambda x$.

For the variation in y,

$$\delta \int_{t_0}^{t_1} \left(\frac{1}{2} m\dot{y}^2 + mgy + \lambda y^2\right) \mathrm{d}t = 0\,,$$

we follow similar reasoning. The kinetic energy term is integrated by parts as before; the potential energy term is $\delta(mgy) = mg\delta y$ and the constraint term is $\delta(\lambda y^2) = 2\lambda y \delta y$. So the y variation leads to,

$$\int_{t_0}^{t_1} (-m\ddot{y} + mg + 2\lambda y)\, \delta y \, \mathrm{d}t\,.$$

Again, the contents of the parentheses must vanish and we have secured these two equations,†

$$m\ddot{x} = 2\lambda x \tag{1}$$
$$m\ddot{y} = mg + 2\lambda y \tag{2}$$

I convert now to angular coordinates for convenience:

$$\begin{aligned} x &= \ell \sin\theta \\ y &= \ell \cos\theta\,, \end{aligned}$$

from which,

$$\begin{aligned} \dot{x} &= \ell\dot{\theta}\cos\theta \\ \dot{y} &= -\ell\dot{\theta}\sin\theta \\ \ddot{x} &= \ell\left(-\dot{\theta}\sin\theta + \ddot{\theta}\cos\theta\right) \\ \ddot{y} &= \ell\left(-\dot{\theta}\cos\theta - \ddot{\theta}\sin\theta\right) \end{aligned}$$

and substituting these into (1) and (2), I obtain,

$$\begin{aligned} m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= 2\lambda\ell\sin\theta \\ -m\ell\ddot{\theta}\sin\theta - m\ell\dot{\theta}^2\cos\theta &= mg + 2\lambda\ell\cos\theta\,. \end{aligned}$$

† Varying with respect to λ gives

$$\int_{t_0}^{t_1} \left(x^2 + y^2 - \ell^2\right) \delta\lambda \, \mathrm{d}t = 0$$

and so for arbitrary variation, $\delta\lambda$, we recover the constraint, $x^2 + y^2 = \ell^2$ as you might have expected.

We multiply the first by $\cos\theta$ and the second by $-\sin\theta$ and add, resulting in,

$$m\ell\ddot{\theta} = -mg\sin\theta\,,$$

which secures the equation of motion for a pendulum,†

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0\,. \qquad (3)$$

For small oscillations, $\sin\theta \approx \theta$ resulting in the well-known equation of motion,

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0\,,$$

whose solution, as is well known, gives the angular frequency,

$$\omega = \sqrt{\frac{g}{\ell}}\,,$$

and hence period,

$$T = 2\pi\sqrt{\frac{\ell}{g}}\,.$$

The solution for the non linear pendulum equation (3) is much harder to obtain and involves an elliptic integral of the first kind.

Now lets find the physical origin of the Lagrange multiplier. Start with the constraint, $x^2 + y^2 = \ell^2$, with ℓ constant. Taking a time derivative twice, we get, firstly,

$$2x\dot{x} + 2y\dot{y} = 0\,, \qquad \text{or,} \qquad x\dot{x} + y\dot{y} = 0\,;$$

and, secondly,

$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + x\ddot{y} = 0\,.$$

We put these into our equations of motion (1) and (2):

$$\dot{x}^2 + \dot{y}^2 + \frac{2\lambda\ell^2}{m} + gy = 0\,,$$

using $x^2 + y^2 = \ell^2$; which, solving for λ results in,

$$\lambda = -\frac{m}{2\ell^2}\left(\dot{x}^2 + \dot{y}^2 + gy\right)\,.$$

In angular coordinates, $\dot{x}^2 + \dot{y}^2 = \ell^2\dot{\theta}^2$, this is,

$$\lambda = -\frac{m}{2\ell}\left(g\cos\theta + \ell\dot{\theta}^2\right)\,,$$

and so,

$$-2\lambda\ell = m\ell\omega^2 + mg\cos\theta\,,$$

since $\dot{\theta} = \omega$ the angular velocity (freqency). The first term in this is the centripetal acceleration, and the second term is the acceleration due to gravity. These two terms add to give the tension in the string, $T = -2\lambda\ell$.

† The mass cancels either side; or if you prefer, Einstein's *equivalence principle* asserts that gravitational and inertia mass are equal.

References

1. Arnold Sommerfeld, *Mechanics* (Academic Press, New York, 1952)
2. Arnold Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press, New York, 1950)
3. Cornelius Lanczos, *The Variational Principles of Mechanics*, (University of Toronto Press, Toronto, 1949)